MATHEMATICS
ESSAY PROBLEMS

Name :

Participant ID :

Country :

Seat Number :

22nd International Mathematics and Science Olympiad
Alor Setar, Kedah, Malaysia
06 October 2025

Instructions:
1. Write your name, country and Participant ID on every page of this booklet.
2. Write your detailed solution in Arabic Numerical or English in the space provided for each question in this booklet. If you need more space for your working solutions, you may use the reverse side of each page, but please write "next side continued" in the last line.
3. There are **13** questions in this paper.
4. Each question is worth 3 marks and partial credits may be awarded.
5. Diagrams are NOT drawn to scale. They are intended only as aids.
6. You have **90** minutes to complete this paper.
7. Use black pen or blue pen or pencil to write your answer.

Do not turn over this page until you are told to do so.

ESSAY PROBLEMS

(with Solution)

1. A four-digit number has the following properties:
   (a) It is a perfect square.
   (b) The first two digits are the same.
   (c) The last two digits are the same.
   List down all possible four-digit number(s) that satisfy the conditions.

Suggested Solution:

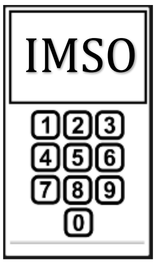
Let $N = aabb$ be the representation of such a number.
$1 \le a \le 9$, $0 \le b \le 9$.
Then $N = 1000a + 100a + 10b + b = 1100a + 11b$
$= 11\ (100a + b)$
Since N is a perfect square and 11 is a factor of N,
$11^2|N \Rightarrow 11|(100a + b) \Rightarrow 11|(a + b) \Rightarrow a + b = 11k$
But $1 \le a + b \le 18 \Rightarrow a + b = 11 \Rightarrow b = 11 - a \Rightarrow b > 0$
The last two non-zero digits of a perfect square where both the digits are equal is only 44.
So, $b = 4$
$\therefore a = 7$
$\therefore N = 7744$ is the only possibility.
$N = 11 \times 704 = 11 \times 11 \times 64 = 88^2$

ANS: 7744

Marking Scheme

- Correct algebraic set-up and factorization (show $\overline{aabb} = 1100a + 11b = 11(100a + b)$, and note the number is divisible by 11. … **(1 point)**
- Use square/divisibility logic to conclude 100a+b must also be divisible by 11, reduce modulo 11 to get $a + b = 11$, and list valid digit-pairs (or state the candidates 2299, 3388, …, 9922). … **(1 point)**
- Identify the correct perfect square among candidates: $7744 = 88^2$, and state the final answer. … **(1 point)**

2. Ahmad wishes to open a door secured by a four-digit code corresponding to the word **IMSO**.



Each letter is assigned a digit subject to the following conditions:
(1) The letters I and O must be odd.
(2) The letters M and S must be even.
(3) All four digits must be distinct.
In how many ways can Ahmad enter the code?

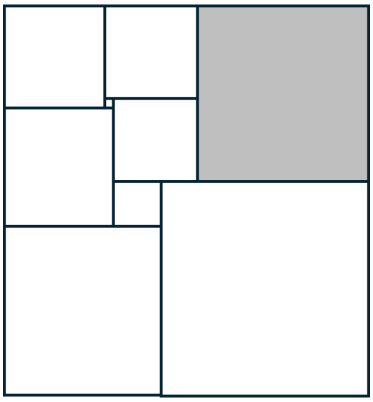
Suggested Solution:
Let the code positions follow the word **IMSO**, where the vowels I and O must each be assigned an odd digit from $\{1,3,5,7,9\}$ and the consonants M and S must each be assigned an even digit from $\{0,2,4,6,8\}$.
Assign distinct odd digits to I and O: $5P2 = 5 \times 4 = 20$ ways.
Assign distinct even digits to M and S: $5P2 = 5 \times 4 = 20$ ways.
Because the odd and even assignments are independent, so total $20 \times 20 = 400$ ways

ANS: 400

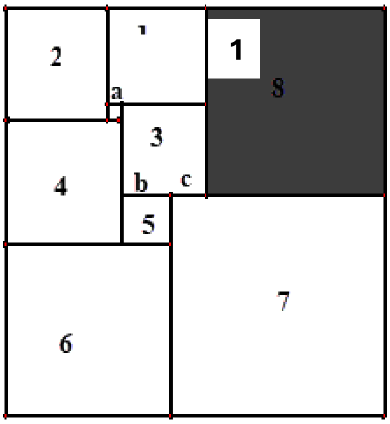
Marking Scheme

- Correct method for assigning distinct odd digits … **(1 point)**
- Correct method for assigning distinct even digits … **(1 point)**
- Correct final answer … **(1 point)**

3. In the diagram below, a rectangle is composed of nine squares. Given that the area of the shaded square is 100 cm^2. What is the area, in cm^2, of the rectangle?



Suggested Solution:
The area of shaded square is 100 cm^2, so the side of it is 10 cm.



Consider, a, b, c are 3 segments in the figure.
- The side of square 8 is: $2(b + c) + a = 10$ ---------------- (1)
- The side of square 1 is: $a + b + c$
- The side of square 2 is: $2a + b + c$
- The side of square 3 is: $b + c$
- The side of square 4 is: $b + (b + c - a) = 2b + c - a$,
but it is also $(2a + b + c) + a = 3a + b + c$
Then: $2b + c - a = 3a + b + c \to b = 4a$ ----------------(2)
- The side of square 5 is: b
- The side of square 6 is: $b + (2b + c - a) = 3b + c - a$
- The side of square 7 is: $4b + c - a$

But, note that, the side of square 7 is also: $c + (2(b+c)+a) = 2b+3c+a$
So: $4b+c-a = 2b+3c+a$, or $2b-2c-2a = 0$, so $b = c+a$.
From (2). we have: $c = 3a$. ----------------(3)
Replace to (1): $2(4a+3a)+a = 10, \Rightarrow a = 2/3$
So the length and the width of the rectangle is:
$7b + 2c - 2a = 28a + 6a - 2a = 32a = 64/3$.
And $6b+3c = 24a + 9a = 33a = 22$
So, the area of the rectangle is: $\frac{64}{3} \times 22 = \mathbf{469\frac{1}{3}}$ **cm²**

ANS: $469\frac{1}{3}$ cm²

Marking Scheme

- Only the answer without explanation **(0 point)**
- Correct explanation to find correct mathematical sentence for the problems **(2 point)**
- Get the final answer **(1 point)**

4. Consider all eight-digit integers of the form $\overline{aaaabbbb}$ which are multiples of 21. List down all possible ordered pairs (*a, b*).

Suggested Solution:
From the question we can say that $aaaabbbb = 11110000a + 1111b$.
Since both 11110000 and 1111 have remainder 1 when divided by 3, so we can say that $a + b$ should be multiple of 3, so $\Rightarrow a + b = 3m$ … (1).
Since both 11110000 and 1111 have remainder 6 and 5 when divided by 7 respectively, so we can say that $6a + 5b = 7(a + b) - (a + 2b)$ should be multiple of 7 and $a + 2b$ also multiple of 7, so $\Rightarrow a + 2b = 7t$ … (2).
From (1) to (2) we can say that $3m + b = 7t$.
The possible $a + b = 3m$ is (3, 6, 9, 12, 15, 18).

- For $a + b = 3$, the possible value of b for which $3m + b = 7t$ is 4, but it's a contradiction since a is a negative number.
- For $a + b = 6$, the possible value of b for which $3m + b = 7t$ is 1, and the possible (*a, b*) is (5,1).
- For $a + b = 9$, the possible value of b for which $3m + b = 7t$ is 5, and the possible (*a, b*) is (4,5).
- For $a + b = 12$, the possible value of b for which $3m + b = 7t$ is 9, and the possible (*a, b*) is (3,9).
- For $a + b = 15$, the possible value of b for which $3m + b = 7t$ is 6, and the possible (*a, b*) is (9,6).
- For $a + b = 18$, the possible value of b for which $3m + b = 7t$ is 3, but it's a contradiction since we will get a is a two-digit number.

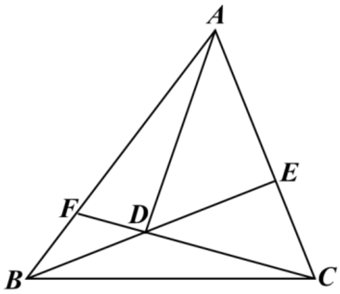
So, all possible values for (*a, b*) are (5, 1), (4, 5), (3, 9) and (9, 6)

ANS: (5, 1), (4, 5), (3, 9) and (9, 6)

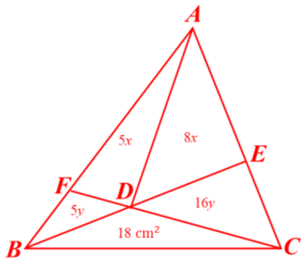
Marking Scheme

- Show that $(a + b)$ is divisible by 3 … **(0.5 point)**
- Show that $(6a + 5b)$ is divisible by 7 … **(0.5 point)**
- Deduce that $3m + b = 7t$ … **(1 point)**
- Find 4 correct answers … **(1 point)**
  Notes: *Find more than 1 but less than 4 correct answers …* ***(0.5 points)***

5. In the diagram below, ABC is a triangle, where E is a point on AC and F is a point on AB. The lines BE and CF intersect at D. If the ratio of the areas of the triangles BDF and CDE is 5:16, the ratio of the areas of the triangles AFD and AED is 5:8 and the area of the triangle BCD is 18 cm^2, what is the area, in cm^2, of the triangle ABC?



Suggested Solution:
Make line AD



Let's denote: $S_{\Delta AFD} = 5x, S_{\Delta AED} = 8x, S_{\Delta BDF} = 5y, S_{\Delta DCE} = 16y$ and $S_{\Delta BCD} = 18$ cm^2.

$$\frac{AF}{BF} = \frac{S_{\Delta AFD}}{S_{\Delta BDF}} \text{ and } \frac{AF}{BF} = \frac{S_{\Delta ACD}}{S_{\Delta BCD}} \Rightarrow \frac{S_{\Delta AFD}}{S_{\Delta BDF}} = \frac{S_{\Delta ACD}}{S_{\Delta BCD}} \Rightarrow \frac{5x}{5y} = \frac{8(x+2y)}{18} \Rightarrow \frac{x}{y} = \frac{4(x+2y)}{9}$$

$$\frac{AE}{CE} = \frac{S_{\Delta AED}}{S_{\Delta DCE}} \text{ and } \frac{AE}{CE} = \frac{S_{\Delta ABD}}{S_{\Delta BCD}} \Rightarrow \frac{S_{\Delta AED}}{S_{\Delta DCE}} = \frac{S_{\Delta ABD}}{S_{\Delta BCD}} \Rightarrow \frac{8x}{16y} = \frac{5(x+y)}{18} \Rightarrow \frac{x}{y} = \frac{5(x+y)}{9}$$

So, we can se that$4(x+2y) = 5(x+y) \Rightarrow x = 3y$

So, $AF : BF = x : y = 3 : 1$, then

$S_{\Delta ACF} : S_{\Delta BCF} = AF : BF \Rightarrow 55y : (18+5y) = 3 : 1 \Rightarrow y = \frac{54}{40}$ and $x = \frac{162}{40}$

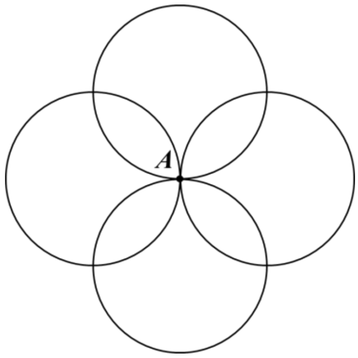
$S_{\Delta ABC} = 13x + 21y + z = 13 \times \frac{162}{40} + 21 \times \frac{54}{40} + 18 = 99$ cm^2

ANS: 99 cm^2

Marking Scheme

- Find $\frac{x}{y} = \frac{4(x+2y)}{9}$ and $\frac{x}{y} = \frac{5(x+y)}{9}$ … **(1 point)**
- Find $x = 3y$ **… (1 point)**
- Get the correct answer $S_{\Delta ABC} = 99$ cm^2 **… (1 point)**

6. Four cyclists start riding at exactly 12:00 noon. Each cyclist rides on a different circular track, where each track is exactly 1 mile long. The four cyclists ride at four different constant speeds, which are 6, 9, 12 and 15 miles per hour, respectively. All four tracks meet at the center (Point A). They all start together at the center and keep riding around their own track. They agree to stop when they have come back to the center together for the sixth time (counting the start at 12:00 noon as the first time). At what time do they stop?



Suggested Solution:

Each time a cyclist finishes one lap he is at the center. So a cyclist is at the center at times that are whole multiples of his lap time. All four are at the center together exactly at times that are common multiples of their lap times. Assume that the four cyclists are A, B, C and D, respectively. We calculate their lap times in seconds (to avoid fraction) as follows;

Cyclist A (6 mph) uses (60 × 60) ÷ 6 = 600 seconds per 1 lap.
Cyclist B (9 mph) uses (60 × 60) ÷ 9 = 400 seconds per 1 lap.
Cyclist C (12 mph) uses (60 × 60) ÷ 12 = 300 seconds per 1 lap.
Cyclist D (15 mph) uses (60 × 60) ÷ 15 = 240 seconds per 1 lap.

We need the smallest positive time that is a multiple of all four lap times. So, we calculate the LCM of 600, 400, 300, and 240 which is 1200 seconds or 20 minutes. We count the starting moment 12:00 noon as the first time they are all together. It means that

1st meet: 12:00:00 (start) — time is 0 minutes after start.
2nd meet: after 20 minutes → 12:20:00 PM.
3rd meet: after 40 minutes → 12:40:00 PM.
4th meet: after 60 minutes → 1:00:00 PM.
5th meet: after 80 minutes → 1:20:00 PM.
6th meet: after 100 minutes → 1:40:00 PM.

Hence, they stop at 1:40:00 PM.

ANS: 1:40:00 PM.

Marking Scheme

- Find the lap time for each cyclist…………………. **(1 point / 0.25 point each)**
- Find the LCM of the 4 time laps………………..… **(1 point)**
- Correct answer……………………………….…….. **(1 point)**
- Correct answer with incorrect solution……………. **(0 point in total)**

7. Let I, M, S and O be digits that are not necessarily different that satisfies

$$\overline{IMSO} = \frac{7}{4} \times \overline{OSMI}$$

How many possible four-digit numbers $\overline{IMSO}$ are there?

Suggested Solution:

$\overline{IMSO} = \frac{7}{4} \times \overline{OSMI}$

$1000I + 100M + 10S + O = \frac{7}{4} \times (10000 + 100S + 10M + I)$

$4000I + 400M + 40S + 4O = 7000O + 700S + 70M + 7I$

$3993I + 330M = 6996O + 660S$ (÷ 33)

$121I + 10M = 212O + 20S$

$10M, 212O$, and $20S$ are definitely even, so $121I$ is also even.

For $I = 2x \leq 8$, we get

$121(2x) + 10M + 212O + 20S \qquad (\div 2)$

$121x + 5M = 106O + 10S$

For $x = O = 1$, we get

$121 + 5M = 106 + 10S$

$15 + 5M = 10S \qquad (\div 5)$

$3 + M = 2S$

$(M, S) \in \{(1,2), (3,3), (5,4), (7,5), (9,6)\}$ there are 5 possibilities

For $x = O = 2$, we get

$242O + 5M = 212 + 10S$

$30 + 5M = 10S \qquad (\div 5)$

$6 + M = 2S$

$(M, S) \in \{(0,3), (2,4), (4,5), (6,6), (8,7)\}$ there are 5 possibilities

For $x = O = 3$, we get

$363 + 5M = 318 + 10S$

$45 + 5M = 10S \qquad (\div 5)$

$9 + M = 2S$

$(M, S) \in \{(1,5), (3,6), (5,7), (7,8), (9,9)\}$ there are 5 possibilities

For $x = O = 4$, we get

$484 + 5M = 424 + 10S$

$60 + 5M = 10S \qquad (\div 5)$

$12 + M = 2S$

$(M, S) \in \{(0,6), (2,7), (4,8), (6,9)\}$ there are 4 possibilities

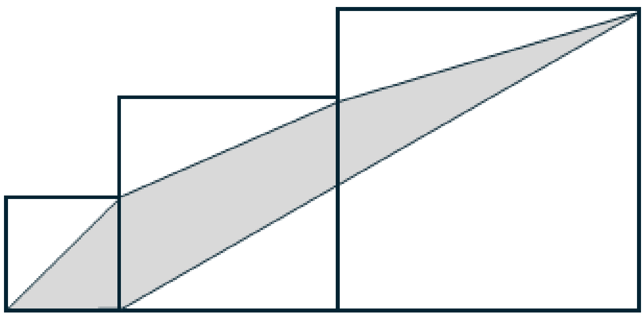
So, there are a total of $5 + 5 + 5 + 4 = 19$ numbers that satisfy the conditions.

ANS: 19

Marking Scheme

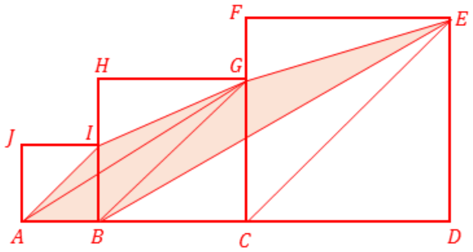
- 1-4 numbers provided with correct proving … **(1 points)**
- 5-9 numbers provided with correct proving … **(1.5 point)**
- 10-14 numbers provided with correct proving … **(2 points)**
- 15-18 numbers provided with correct proving … **(2.5 points)**
- All 19 numbers provided with correct proving … **(3 points)**

8. In the diagram below, three squares with side lengths in the ratio of 2 : 3 : 5 are placed next to each other. If the area of the shaded region is 152 cm^2, then what is the perimeter, in cm, of the diagram?



Suggested Solution:

Look at the figure below!



Let $AB = 2x$, $BC = 3x$ and $CD = 5x$, then area of $ABIJ = 4x^2$, area of $BCGH = 9x^2$ and area of $CDEF = 25x^2$.

After we added some lines and named the vertices, now we know that $AI \parallel BG \parallel CE$.

- Because $AI \parallel BG$ then area of AGI equals to area of $ABI = \frac{1}{2} \times$ area of $ABIJ = 2x^2$.
- Because $BG \parallel CE$ then area of BEG equals to area of $BCG = \frac{1}{2} \times$ area of $BCGH = \frac{9}{2}x^2$.
- Area of $ABG = \frac{1}{2} \times AB \times BC = \frac{1}{2} \times 2x \times 3x = 3x^2$.

The area of shaded region = area of AIG + area of BEG + area of ABG

$\Rightarrow 152\ \text{cm}^2 = 2x^2 + \frac{9}{2}x^2 + 3x^2 = \frac{19}{2}x^2$

$\Rightarrow x^2 = \frac{152 \times 2}{19} = 16 \Rightarrow x = 4.$

The perimeter of the shape

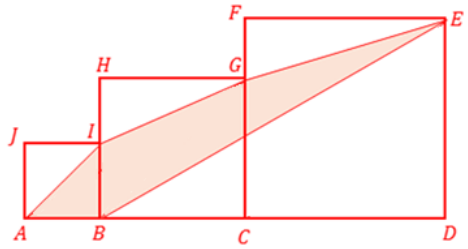
$= 2\times(AB + BC + CD + DE) = 2\times(2x + 3x + 5x + 5x) = 30x = 30 \times 4 =$ **120 cm**.

Marking Scheme

- Find area of *ABIJ and BCGH* … **(0,5 point)**
- Find area of *ABG* … **(0,5 point)**
- Find $x = 4$ … **(1 point)**
- Find perimeter of the shape … **(1 point)**

Suggested Solution 2:

Look at the figure below!



Let $AB = 2x$, $BC = 3x$ and $CD = 5x$, then area of $ABIJ = 4x^2$, area of $BCGH = 9x^2$ and area of $CDEF = 25x^2$.

The area of shaded region = (area of $ABIJ$ + area of $BCGH$ + area of $CDEF$) – (area of AIJ + area of GHI + area of EFG + area of BDE.

$\Rightarrow 152\ \text{cm}^2 = (4x^2 + 9x^2 + 25x^2) - \left(\frac{4}{2}x^2 + \frac{3}{2}x^2 + \frac{10}{2}x^2 + \frac{40}{2}x^2\right)$

$\Rightarrow 152\ \text{cm}^2 = (38x^2) - \left(\frac{57}{2}x^2\right) = \left(\frac{76}{2}x^2\right) - \left(\frac{57}{2}x^2\right)$

$\Rightarrow 152\ \text{cm}^2 = \left(\frac{19}{2}x^2\right)$

$\Rightarrow x^2 = \frac{152 \times 2}{19} = 16 \Rightarrow x = 4.$

The perimeter of the shape

$= 2\times(AB + BC + CD + DE) = 2\times(2x + 3x + 5x + 5x) = 30x = 30 \times 4 =$ **120 cm**.

ANS: 120 cm.

Marking Scheme

- Find formula of the area of shaded $\left[152\ \text{cm}^2 = \left(\frac{19}{2}x^2\right)\right]$ … **(1 point)**
- Find $x = 4$ … **(1 point)**
- Find perimeter of the shape … **(1 point)**

9. In a video game, you start the game with 10 gold coins, 10 wooden coins, 10 stone coins, 0 silver coins and 0 diamond coins.

10 Gold coins **0** Silver coins **10** Wooden coins **10** Stone coins **0** Diamond coins

There are four trading options that are available:

a.) Trading gold coin(s) to a silver coin:
The cost increases by 1 gold coin each time you do a trade.
- First trade: exchange 1 gold coin → receive 1 silver coin.
- Second trade: exchange 2 gold coins → receive 1 silver coin.
- Third trade: exchange 3 gold coins → receive 1 silver coin, and so on.

b.) Trading wooden coin(s) to silver coins:
The cost increases by 1 wooden coin each time you do a trade.
- First trade: exchange 3 wooden coins → receive 2 silver coins.
- Second trade: exchange 4 wooden coins → receive 2 silver coins.
- Third trade: exchange 5 wooden coins → receive 2 silver coins, and so on.

c.) Trading stone coins(s) to a diamond coin:
The cost increases by 1 stone each time you do a trade.
- First trade: exchange 2 stone coins → receive 1 diamond coin.
- Second trade: exchange 3 stone coins → receive 1 diamond coin.
- Third trade: exchange 4 stone coins → receive 1 diamond coin, and so on.

d.) Trading a silver coin to a diamond coin:
Fixed rate: exchange 1 silver coin → receive 1 diamond coin.

Some additional trading conditions:

- You may perform trades in any order and any number of times, provided you have enough coins at the moment of the trade.
- If you do not have enough coins to perform the required condition for a trade, then you cannot perform that trade. (You cannot partially perform a trade).
- The counters for each of the different trade types are independent with each other.

What is the maximum number of diamond coins you can obtain? Give a full and detailed explanation and describe the general strategy that achieves the maximum.

Suggested Solution:

Based on the 4 trading rules, we can obtain diamonds from stone and silver. Before that, gold and wood can be trade for silver. Therefore, we can trade as follows;

Step	Trade Type	Cost	Gain	Gold	Wood	Stone	Silver	Diamond
1	Gold → Silver	1 Gold	+1 Silver	9	10	10	1	0
2	Gold → Silver	2 Gold	+1 Silver	7	10	10	2	0
3	Gold → Silver	3 Gold	+1 Silver	4	10	10	3	0
4	Gold → Silver	4 Gold	+1 Silver	0	10	10	4	0
5	Wood → Silver	3 Wood	+2 Silver	0	7	10	6	0
6	Wood → Silver	4 Wood	+2 Silver	0	3	10	8	0
7	Silver → Diamond	1 Silver	+1 Diamond	0	3	10	7	1

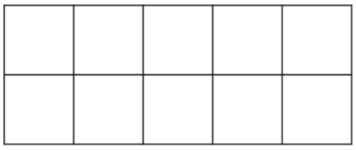
Step	Trade Type	Cost	Gain	Gold	Wood	Stone	Silver	Diamond
8	Silver → Diamond	1 Silver	+1 Diamond	0	3	10	6	2
9	Silver → Diamond	1 Silver	+1 Diamond	0	3	10	5	3
10	Silver → Diamond	1 Silver	+1 Diamond	0	3	10	4	4
11	Silver → Diamond	1 Silver	+1 Diamond	0	3	10	3	5
12	Silver → Diamond	1 Silver	+1 Diamond	0	3	10	2	6
13	Silver → Diamond	1 Silver	+1 Diamond	0	3	10	1	7
14	Silver → Diamond	1 Silver	+1 Diamond	0	3	10	0	8
15	Stone → Diamond	2 Stone	+1 Diamond	0	3	8	0	9
16	Stone → Diamond	3 Stone	+1 Diamond	0	3	5	0	10
17	Stone → Diamond	4 Stone	+1 Diamond	0	3	1	0	11

ANS: 11 diamonds

Marking Scheme

- Trade 10 gold for 4 silver……………………….…. **(0.5 point)**
- Trade 7 wood for 4 silver…………………….……. **(0.5 point)**
- Trade 8 silver for 8 diamond……………………… **(0.5 point)**
- Trade 9 stone for 3 diamond……………………… **(0.5 point)**
- Correct answer…………………………………….. **(1 point)**
- Correct answer with incorrect solution……………. **(0 point in total)**

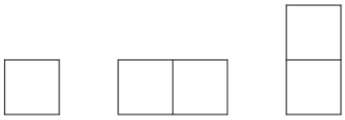
10. The diagram below shows a grid with 2 rows and 5 columns.



2 × 5 grid

The grid must be completely covered using the following types of tiles namely:

- a 1-unit tile, which occupies one space (1 × 1) or
- a 2-unit tile, which occupies two spaces and may be placed either horizontally (1 × 2) or vertically (2 × 1).

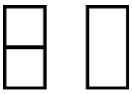


1 × 1 tile 1 × 2 tile 2 × 1 tile

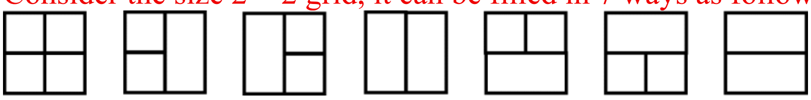
How many different ways are there to completely cover the entire 2 × 5 grid without overlapping?

Suggested Solution:

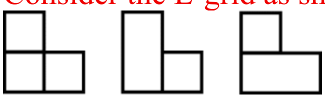
Consider the size 2 × 1 grid, it can be filled in 2 ways as follows;



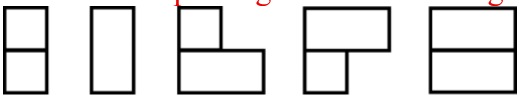
Consider the size 2 × 2 grid, it can be filled in 7 ways as follows;



Consider the L-grid as shown, it can be filled in 3 ways as follows;



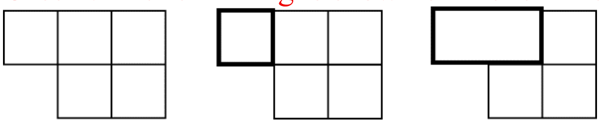
Consider the parking lot of size 2×3 grid, there are 5 cases of starter as follows;



Case 1 Case 2 Case 3 Case 4 Case 5

Case 1: The remaining is 2×2 grid which can be filled in 7 ways.
Case 2: The remaining is 2×2 grid which can be filled in 7 ways.
Case 3: The remaining is L-grid which can be filled in 3 ways.
Case 4: The remaining is L-grid which can be filled in 3 ways.
Case 5: The remaining is 2×1 grid which can be filled in 2 ways.
Hence, the parking lot of size 2×3 grid can be filled in $7 + 7 + 3 + 3 + 2 = 22$ ways.

Consider the size 2×4 grid with the same 5 cases of starter.
Case 1: The remaining is 2×3 grid which can be filled in 22 ways.
Case 2: The remaining is 2×3 grid which can be filled in 22 ways.
Case 3: The remaining is shown on the left and we can continue in 2 cases.



Remaining grid Case 3.1 Case 3.2

From both subcases, the remaining can be filled in $7 + 3 = 10$ ways.
Case 4: This case is similar to Case 3. It can be filled in 10 ways.
Case 5: The remaining is 2×2 grid which can be filled in 7 ways.
Hence, the parking lot of size 2×4 grid can be filled in
$22 + 22 + 10 + 10 + 7 = 71$ ways.
Consider the size 2×5 grid with the same 5 cases of starter.
Case 1: The remaining is 2×4 grid which can be filled in 71 ways.
Case 2: The remaining is 2×4 grid which can be filled in 71 ways.
Case 3: The remaining is shown on the left and we can continue in 2 cases.

Remaining grid Case 3.1 Case 3.2

From both subcases, the remaining can be filled in $22 + 10 = 32$ ways.
Case 4: This case is similar to Case 3. It can be filled in 32 ways.
Case 5: The remaining is 2×3 grid which can be filled in 22 ways.
Hence, the size 2×5 grid can be filled in
$71 + 71 + 32 + 32 + 22 = 228$ ways.

ANS: 228

Marking Scheme

- Correctly count covering a 2 x1 grid … 0.25 point
- Correctly count covering a 2x2 grid …. 0.25 point
- Correctly count covering a 2x3 grid … 1 point
- Correctly count covering a 2x4 grid … 1 point
- Correctly count covering a 2x5 grid … 0.5 point
- **Correctly completing the counting …. 0.5** point
- **Stating the correct answer without any reasoning …** 0 points

11. For each positive integer n, let $S(n)$ be the sum of its digits and $P(n)$ be the product of its digits. For example, if $n = 2025$,

$$S(2025) = 2 + 0 + 2 + 5 = 9 \text{ and } P(2025) = 2 \times 0 \times 2 \times 5 = 0.$$

List down all the possible three-digit numbers n that satisfy $n = S(n) \times P(n)$.

Suggested Solution:

Assume that $n = \overline{abc}$.

Set $x = \min(a, b, c)$, $z = \max(a, b, c)$ and $y = a + b + c - (x + z)$ is the remaining digit of n. Then $n = abc(a + b + c) = xyz(x + y + z)$.

We have:

- None of the digits of n is zero. So, if one of the digit of n is 5, the others must be odd. In this case, we have $c = 5$.
- The case $x = y = z$ is impossible, since it implies that $n = 111x = 3x^4$, or $x^3 = 37$, which is a contradiction.
- If $x = y = 1$, then $n = z(2 + z) \le 9 \cdot 11 = 99$, which is also a contradiction. So $1 \le x, 2 \le y \le z$ and $5 \le x + y + z \le 26$.

Since $n = 100a + 10b + c = 9(11a + b) + (a + b + c) = abc(a + b + c)$, we have

$$9(11a + b) = (abc - 1)(a + b + c).$$

So both sides are divisible by 9 and at least one factor of the right hand side must be divisible by 3.

We consider the following cases:

- **Case 1:** $(abc - 1) = (xyz - 1)$ is divisible by 3, or equivalently, $xyz = 1 \ (mod\ 3)$. Then either:
  - $x = y = z = 1 \ (mod\ 3)$.

  Hence,

x	y	z	$xyz(x + y + z)$	Validity
1	4	4	144	OK
1	4	7	Last digit is 6	Invalid
1	7	7	Last digit is 5	Invalid
4	4	7	Last digit is 5	Invalid
4	7	7	Last digit is 8	Invalid

  - Or one digit of n is equal to $1 \ (mod\ 3)$ and the remaining digits are of the form $-1 \ (mod\ 3)$. Then $n = x + y + z = -1 \ (mod\ 3)$.

  We label the digits of n as $u = 1 \ (mod\ 3) \in \{1, 4, 7\}$ and $v = t = -1 \ (mod\ 3) \in \{2, 5, 8\}$. Assume that $v \le t$. Recall that if a digit of n is 5, then all digits of n are odd.

u	v	t	$uvt(u + v + t)$	Validity
1	2	2	20	Invalid
1	2	8	Last digit is 6	Invalid
1	5	5	275	Invalid
1	8	8	>1000	Invalid
4	2	2	Last digit is 8	Invalid
4	2	8	Last digit is 6	Invalid
4	8	8	Last digit is 0	Invalid
7	2	2	Last digit is 8	Invalid
7	2	8	Last digit is 4	Invalid
7	5	5	>1000	Invalid
7	8	8	>1000	Invalid

  So for Case 1, the only valid number is 144.

- **Case 2**: $(xyz - 1)$ is not divisible by 3. Then $x + y + z$ must be divisible by 9.
  - First we will show that $x + y + z = 18$ is impossible.
    Let assume that $x + y + z = 18$. Since $x + y + z > 3x$ and $n = 18\ xyz < 990$, we must have $x \leq 5, 7 \leq z \leq 9, x + y \geq 9$ (so $y \geq 5$) and $xyz < \frac{990}{18} = 55$.
    If $x = 1$, then $y = 8, z = 9$, which is invalid since $xyz = 72 > 55$.
    If $x \geq 2$, then $xyz \geq 2 \cdot 5 \cdot 7 = 70 > 55$.
    In either case, we have a contradiction, which means that the assumption $x + y + z = 18$ does not hold.
  - Now the only possibility is $x + y + z = 9$.
    In this case, $1 \leq x \leq 2,\ 2 \leq y \leq 4,\ 3 \leq z \leq 6$.
    If $x = 2$, then $y = 3,\ z = 4$, but $234 \neq 2 \cdot 3 \cdot 4 \cdot 9$.
    If $x = 1$, then either $y = 2, z = 6$ or $y = 3, z = 5$ (the case $x = 1, y = z = 4$ has been considered above). We can verify that the only valid possibility is $x = 1, y = 3,\ z = 5$ with $n = 135$.

So, the values of n are 135 and 144.

ANS: 135 and 144

Marking Scheme

- List conditions for the digits of n … **(0.5 point)**
- Show that $(xyz - 1)(x + y + z)$ is divisible by 9 … **(0.5 point)**
- Analyze Case 1 and find $n = 144$ … **(1 point)**
- Analyze Case 2 and find $n = 135$ … **(1 point)**

12. Twenty students participated in a chess tournament, where each student played every other student exactly twice.
A student earns 3 points for a win, 1 point for a draw and 0 points for a loss.
What is the maximum number of students that can have at least 90 points at the end of the tournament?

Suggested Solution:

There are 20*19/2 = 190 pairs of players
As each player plays 2 games there are 380 games altogether. Therefore, the max number of points is 380*3 = 1140 points (as we have 3 points if there is a win and 2 points it there is a draw, if we want to maximize the number of points then we shall have all wins).
Each player plays 2*19=38 games and the max points he can win is 114 points.
Assume there are k players scoring 90 or higher, thus these k players will win at least 90k points.
The remaining points are 1140 – 90k and these are the points distributed among the 20-k players, thus it is a positive number.
1140-90k shall be positive thus k<=12

Let's try k=12

Consider the games:
Each of the top 12 plays 38 games: 19 against other top12 (each pair plays twice) and 20 against the bottom8 (each pair plays twice?
Actually, each top player plays $2\times11=22$ games against other top 12, and $2\times8=16$ games against bottom 8. To give each top player exactly 90 points, we can design outcomes:
For games between top 12 and bottom 8: let every top player win both games against every bottom player. Then each top player gets $16\times3=48$ points from these games.
They need $90-48=42$ points from the 22 games against other top12.
Total points needed from games among top 12: $12\times42=504$ points.
Number of games among top12: $2\times66=132$ games.

If all are wins/losses, each gives 3 points, total = 132×3=396, still less than 504.
So, it is impossible to get 504 points from 132 games. The maximum from 132 games is 132×3=396, which is less than 504.
So k=12 is impossible.

Let's try k=11

Total points needed for top11: 11×90=990.
Each top player plays
Against other top11:
2×10=20 games.

Against bottom 9:
2×9=18 games.
If each top player wins all against bottom 9: 18×3=54 points.
They need 90−54=36 points from 20 games against other top11.
Total points needed from inner games: 11×36=396.
Number of inner games: 2×55=110 games.
Maximum points from these games: 110×3=330, which is less than 396.
Still not possible.

Let's try k=10

Top10: each needs 90.
Vs bottom10: 2×10=20 games. Win all: 20×3=60.
Need 30 from games vs other top10: 2×9=182×9=18 games.
Total needed from inner: 10×30=300
Number of inner games: 2×45=90.
Max points from inner games: 90×3=270<300.
Not possible.

Let's try k=9

Top 9: each needs 90.
Vs bottom11: 2×11=22 games. Win all: 22×3=66.
Need 24 from games vs other top9: 2×8=16 games.
Total needed: 9×24=216.
Inner games: 2×36=72.
Max points: 72×3=216.
So, we can have 9 players with exactly 90 points:

- Each wins both games against all 11 bottom players: 22wins=66 points.
- In the games among the top9, we need to distribute exactly 216 points in 72 games. This is possible if every game is decisive (win/loss), and the points are arranged so that each of the 9 gets exactly 8 wins and 8 losses in these 16 games? Actually, each plays 16 games, and we need each to get 24 points, i.e., 8 wins (since 8*3=24).

One can distribute this easily if each player wins one of the games to everyone else and loses the other one.

ANS: 9 students

Marking Scheme

- Get max number of games … **(0.5 point)**
- Get max number of points … **(0.5 point)**
- Deduce that k<=12 … **(0.5 point)**
- Show one possible way for k=9 is possible … **(1 point)**
- Conclude that k=9 is possible … (**0.5 point)**
- Conclude that k=9 is possible (without any reasoning) … **(0 point)**

13. Let a, b, and c be distinct positive integers satisfying $a^b \times b^a = (a \times b)^c$. What is the smallest possible value of $a + b + c$ across all valid triples (a, b, c)?

Suggested Solution:

$a^b b^a = (ab)^c \Rightarrow a^{b-c} = b^{c-a}$.

Since a, b, c are distinct, and WLOG $a < b$ then $b = a^m$. for integer $m > 1$.

Then $a^{b-c} = b^{c-a} \Rightarrow a^{b-c} = a^{m(c-a)}$, and $b - c = m(c - a) \Rightarrow a^m - c = m(c - a)$

$\Rightarrow a^m + am = mc + c \Rightarrow c = \frac{a^m - am}{1 + m}$.

To make the smallest $a + b + c$, so, we try to make the smallest a.

- for $a = 2$

  $c = \frac{2^m + 2m}{1+m} = \frac{2(2^{m-1} + m)}{1+m}$, m should be odd that larger than 1, the smallest possible c when m =5, c =7 and $b = 2^5 = 32$.

  $a + b + c = 2 + 32 + 7 = 41$
- for $a = 3$

  $c = \frac{3^m + 3m}{1+m} = \frac{3(3^{m-1} + m)}{1+m}$, m should have remainder 2 when divided by 3, the smallest m is 2, so $c = \frac{3(3^{2-1} + 2)}{1+2} = 5$, then $b = 3^2 = 9$.

  $a + b + c = 3 + 9 + 5 = 17$.
- for $a = 4$

  $c = \frac{4^m + 4m}{1+m} = \frac{4(4^{m-1} + m)}{1+m}$, m should be odd that larger than 1, the smallest m is 3, so $c = \frac{4(4^{3-1} + 3)}{1+3} = 19$, then $b = 4^3 = 64$.

  $a + b + c = 4 + 64 + 19 = 87$.
- for $a > 4$, the sum for $a + b + c$ will be > 17

  So, the smallest possible value of $a + b + c = 3 + 9 + 5 =$ **17**.

ANS: 17

Marking Scheme

- Express c in terms of a and m or equivalent … **(1 point)**
  - **If the student noted that $a = b^m$ … (0.5 points)**
- For reviewing each of the cases when a = 2,3,4 and obtaining the solutions for a, b, and c … **(0.5 points each)**
- Correct answer … **(0.5 points)**
- *If the correct answer was given with some sort of educated guessing – (1 point)*